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AXISYMMETRIC FLEXURE OF AN INFINITE PLATE RESTING ON A FINITELY DEFORMED INCOMPRESSIBLE ELASTIC HALFSPACE

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Abstract—This paper investigates the axisymmetric flexural behaviour of an infinite elastic plate resting on an isotropic incompressible elastic halfspace which is initially deformed by a state of finite radial extension or compression. The small axisymmetric flexural deformations of the infinite plate are due to forces which act normal to the plane of radial deformation. The basic problem is of interest in connection with geomechanics problems which deal with interaction analysis of the earth's crustal plate and the underlying mantle.

1. INTRODUCTION

The general theory of small deformations superposed on a finite deformation of an isotropic elastic material was developed by Green, Rivlin and Shield[1]. This theory provides exact solutions for problems of infinitesimal strains superimposed on an assigned and exact large initial deformation. At the same time this theory offers a natural extension to the classical theory of linear elasticity which takes into consideration the effects of initial stress. Detailed accounts of the method of Green *et al.*[1] together with references to further studies involving small deformations superposed on large, are given by Green and Zerna[2] and Eringen and Suhubi[3].

The present paper is concerned with the axisymmetric flexure of an infinite elastic plate resting on an incompressible elastic halfspace which is initially subjected to a uniform finite radial deformation. Formal analytical results are presented for the deflections and contact stresses in an infinite plate which is subjected to an arbitrary axisymmetric external load. Specific numerical results are presented for the deflection and contact stress in an infinite plate at the point of application of a concentrated force. These numerical results are restricted to an incompressible elastic material with a strain energy function of the Mooney–Rivlin[4] type.

The basic problem discussed here has potential applications in the field of Geomechanics. For short term loads of a geological nature (10^3–10^4 yr) the earth's lithosphere (the crustal plate) and the asthenosphere (the mantle) are usually modelled as a thin elastic plate and a dense fluid substratum respectively (see Nadai[5]; Brotchie and Silvester[6]; Walcott[7]). The flexural rigidity of the lithosphere is then deduced from observations of the wavelength and amplitude of bending in the vicinity of supercrustal loads. The thickness of the earth's lithosphere in the vicinity of these loads can then be inferred from the computed values of the crustal rigidity. The accuracy of such estimates will be dependent upon, among other factors, the type of model adopted to represent the asthenosphere. This paper extends the analytical study of the problem of crustal flexure to take into account the influence of factors such as rigidity and state of stress of the asthenosphere. Admittedly, there is no clear evidence to suggest that the representation of the mantle behaviour by an elastic material with a strain energy function of the Mooney–Rivlin type is in anyway adequate; the analysis presented here can be extended to include other forms of strain energy functions. However, by introducing a relatively simple form of a strain energy function consistent with incompressible elastic behaviour it becomes possible to examine the influence of the rigidity and the initial stress effects in the mantle, which are altogether neglected in the existing treatments (see, e.g. Cathles[8]).

2. NOTATION AND FORMULAE

The method of analysis adopted in this paper is essentially that outlined by Green and Zerna[2]. The relevant results are briefly summarized for completeness.

The points in the isotropic incompressible elastic body are defined by a general curvilinear coordinate system $\theta_i(\theta_1 = r;\ \theta_2 = \theta;\ \theta_3 = z)$ which moves with the body as it deforms. The covariant and contravariant metric tensors corresponding to the undeformed and deformed states of the body are denoted by g_{ij}, G_{ij} and g^{ij}, G^{ij} respectively. The strain invariants I_1, I_2 and I_3 are given by

$$I_1 = g^{rs}G_{rs};\quad I_2 = g_{rs}G^{rs};\quad I_3 = |G_{ij}|/|g_{kl}| \tag{1}$$

where a repeated index denotes contraction over the dummy index. The requirement $I_3 = 1$, ensures that the deformations are isochoric. In the case of incompressible isotropic elastic materials, the contravariant stress tensor τ^{ij}, measured per unit area of the deformed body and referred to the θ_i coordinates of the deformed body, is given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij} \tag{2}$$

where

$$B^{ij} = I_1 g^{ij} - g^{ir}G^{js}G_{rs} \tag{3a}$$

$$\Phi = 2\frac{\partial W}{\partial I_1};\qquad \Psi = 2\frac{\partial W}{\partial I_2} \tag{3b}$$

and $W = W(I_1, I_2)$ is the strain energy function per unit volume of the material. The scalar pressure p is a function of position, which has to be determined by satisfying the boundary conditions of the particular problem.

In connection with the solution of the title problem, we restrict our attention to the halfspace region which is subjected to a finite radial stretch μ, with zero surface traction on the bounding plane. For this particular finite deformation problem, the non-zero components of the contravariant stress tensor are

$$\tau^{11} = r^2\tau^{22} = \left(\mu^2 - \frac{1}{\mu^4}\right)(\Phi + \mu^2\Psi). \tag{4}$$

We now superpose on the finitely deformed halfspace a further infinitesimal axially symmetric motion characterized by the following displacement field:

$$u_r = u(r, z);\quad u_\theta = 0;\quad u_z = w(r, a). \tag{5}$$

For isochoric motions of the material, this displacement field should satisfy the incompressibility condition

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{6}$$

The components of the stress tensor τ'^{ij} governing the superposed motion can be expressed as (see Green and Zerna [2])

$$\begin{aligned}
\tau'^{11} &= p' + \alpha_1\frac{\partial w}{\partial z} + \alpha_2\frac{u}{r}\\
r^2\tau'^{22} &= p' + \alpha_1\frac{\partial w}{\partial z} + \alpha_2\frac{\partial u}{\partial r}\\
\tau'^{33} &= p' + \alpha_3\frac{\partial w}{\partial z}\\
\tau'^{13} &= \alpha_4\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\alpha_1 &= \zeta_1 - \frac{2c}{\mu^4};\quad \alpha_2 = \frac{2}{\mu^4}(\Phi + \mu^8\Psi) - \frac{2c}{\mu^4}\\
\alpha_3 &= \zeta_2 + \frac{2c}{\mu^4};\quad \alpha_4 = \frac{c}{\mu^4}
\end{aligned} \tag{8a}$$

and

$$c = (\Phi + \mu^2\Psi)$$

$$\zeta_1 = 2A\left(\frac{1}{\mu^2} - \mu^4\right) + 2B\left(\frac{1}{\mu^4} - \mu^8\right) + 2F\left(1 + \frac{1}{\mu^6} - 2\mu^6\right)$$

$$\zeta_2 = 2\left(\frac{1}{\mu^2} - \mu^4\right)\left\{\frac{A}{\mu^6} + \frac{2B}{\mu^2} + \frac{3F}{\mu^4}\right\} \tag{8b}$$

$$A = 2\frac{\partial^2 W}{\partial I_1^2}; \quad B = \frac{\partial^2 W}{\partial I_2^2}; \quad F = 2\frac{\partial^2 W}{\partial I_1 \partial I_2}.$$

Also, in the particular case where the small axisymmetric deformation is superposed on an initial homogeneous finite radial deformation and, in the absence of body forces, the equations of equilibrium for the superposed stress field τ'^{ij} reduces to

$$\frac{\partial \tau'^{11}}{\partial r} + \frac{\partial \tau'^{13}}{\partial z} + \frac{\tau'^{11}}{r} - r\tau'^{22} - \tau^{11}\frac{\partial^2 w}{\partial r \partial z} = 0$$

$$\frac{\partial \tau'^{13}}{\partial r} + \frac{\partial \tau'^{33}}{\partial z} + \frac{\tau'^{13}}{r} + \tau^{11}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right\} = 0. \tag{9}$$

Similarly the traction boundary conditions corresponding to the superposed field are given by

$$\tau'^{ij}n_i + \tau^{ij}n'_i = P'^j \tag{10}$$

where n_i is the covariant component of the unit normal referred to a surface in the finitely deformed body; n'_i and P'^j are the covariant component of the unit normal and the contravariant component of the surface force vector referred to a bounding surface in the finitely deformed state.

3. DISPLACEMENT FUNCTIONS

As pointed out by Green *et al.*[1] and Woo and Shield[9], the solution of the equations governing the superposed deformation is facilitated by the use of potential function techniques which have been developed for the analysis of the classical problem in anisotropic elasticity theory. Briefly, the solution to the superposed displacements (5) can be expressed in terms of two functions $\phi_n (n = 1, 2)$ in the following form

$$u = \frac{\partial \phi_1}{\partial r} + \frac{\partial \phi_2}{\partial r}; \quad w = k_1\frac{\partial \phi_1}{\partial z} + k_2\frac{\partial \phi_2}{\partial z} \tag{11}$$

where k_1 and k_2 are roots of the equation

$$k^2\{\alpha_4 + \tau^{11}\} + k\{\alpha_1 - \alpha_3 + 2\alpha_4 - \tau^{11}\} + \alpha_4 = 0. \tag{12}$$

The functions ϕ_n are solutions of the equations

$$\left\{\tilde{\nabla}^2\phi_n + k_n\frac{\partial^2 \phi_n}{\partial z^2}\right\} = 0; \quad (n = 1, 2) \tag{13a}$$

where

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{13b}$$

The hydrostatic pressure p', indeterminate to within an arbitrary constant value, is given by either of the following equations

$$p' = \begin{cases} \{-\alpha_4 - k_1(\alpha_1 + \alpha_4 - \tau^{11})\}\dfrac{\partial^2\phi_1}{\partial z^2} + \{-k_2(\alpha_3 - \alpha_4) + k_2^2(\alpha_4 + \tau^{11})\}\dfrac{\partial^2\phi_2}{\partial z^2} \\ \{-k_1(\alpha_3 - \alpha_4) + k_1^2(\alpha_4 + \tau^{11})\}\dfrac{\partial^2\phi_1}{\partial z^2} + \{-\alpha_1 - k_2(\alpha_1 + \alpha_4 - \tau^{11})\}\dfrac{\partial^2\phi_2}{\partial z^2} \end{cases} \tag{14}$$

4. AXISYMMETRIC LOADING OF THE ELASTIC HALFSPACE

For future reference, we apply the preceeding theory to develop the solution to the problem of an initially deformed halfspace which is subjected to an axisymmetric normal traction $g(r)$. The method solution employed here is an adaptation of the Hankel transform method of solution of axisymmetric problems in classical elasticity as developed by Sneddon[10]. The zeroth order Hankel transform of the function $g(r)$ is defined as

$$\bar{g}^0(\xi) = \mathcal{H}_0\{g(r); \xi\} \tag{15a}$$

where the operator $\mathcal{H}_0$ is defined by the equation

$$\mathcal{H}_0\{g(r); \xi\} = \int_0^\infty rg(r)J_0(\xi r/a)\,\mathrm{d}r \tag{15b}$$

and a is a typical length parameter in the problem. The Hankel inversion theorem yields the inverse of the Hankel transform

$$g(r) = \frac{1}{a^2}\int_0^\infty \xi\bar{g}^0(\xi)J_0(\xi r/a)\,\mathrm{d}\xi. \tag{15c}$$

Operating on equation (13a) with the zeroth order Hankel transform, we obtain the following pair of second-order ordinary differential equations for the transformed stress functions $\bar{\phi}_n^0(\xi, z)$;

$$\left\{k_1\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{\xi^2}{a^2}\right\}\bar{\phi}_1^0(\xi, z) = 0; \quad \left\{k_2\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{\xi^2}{a^2}\right\}\bar{\phi}_2^0(\xi, z) = 0. \tag{16}$$

The solutions of (16) appropriate for the halfspace problem are given by

$$\bar{\phi}_1^0(\xi, z) = R_1\,\mathrm{e}^{-(\xi z/a\sqrt{k_1})} + R_2\,\mathrm{e}^{-(\xi z/a\sqrt{k_2})} \tag{17}$$

where the coefficients of the positive exponential terms are taken as zero so that the stresses and displacements in the halfspace remain bounded at $z \to \infty$. The arbitrary constants R_1 and R_2 are determined from the traction boundary conditions

$$\tau'^{33}(r, 0) = g(r); \quad \tau'^{13}(r, 0) = 0. \tag{18}$$

From the above conditions we obtain

$$R_1 = -R_2\left(\frac{1+k_2}{1+k_1}\right)\sqrt{\left(\frac{k_1}{k_2}\right)}; \quad R_2 = -\frac{\bar{g}^0 a^2}{\xi^2\theta} \tag{19}$$

$$\theta = \left[\left(\frac{1+k_2}{1+k_1}\right)\sqrt{\left(\frac{k_1}{k_2}\right)}\left\{\frac{\alpha_4}{k_1} - \alpha_3 + \alpha_1 + \alpha_4 - \tau^{11}\right\} + \alpha_4 + k_2(\alpha_4 + \tau^{11})\right]$$

and $\bar{g}^0(\xi)$ is the zeroth order Hankel transform of the external normal traction. The displacement and stress components of the superposed deformation can be obtained by making use of the eqns (7), (11), (17) and the inversion theorem (15c).

For example, in the particular case when $g(r)$ corresponds to a concentrated force of magnitude P applied at the origin of coordinates, $\bar{g}^0(\xi) = P/2\pi$; by making use of (11), (17) and (19), the transformed expression for the displacement $w(r, z)$ is given by

$$\bar{w}^0(\xi, z) = \frac{\bar{g}^0(\xi)a}{\xi\theta\sqrt{k_2}}\left[k_2\, e^{-(\xi z/a\sqrt{k_2})} - \left(\frac{1+k_2}{1+k_1}\right) k_1\, e^{-(\xi z/a\sqrt{k_1})}\right]. \tag{20}$$

Using the inversion theorem (15c) we obtain the surface displacement of the initially deformed halfspace which is subjected to a superposed concentrated force P as

$$w(r, z) = \frac{P}{2\pi\theta}\left[\frac{k_2}{\{k_2r^2 + z^2\}^{1/2}} - \left(\frac{1+k_2}{1+k_1}\right)\sqrt{\left(\frac{k_1}{k_2}\right)}\frac{k_1}{\{k_1r^2 + z^2\}^{1/2}}\right]. \tag{21}$$

In the particular case of an incompressible elastic material with a strain energy function of the Mooney–Rivlin type

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{22}$$

in which C_1 and C_2 are constants, we have $k_1 = 1$ and $k_2 = 1/\mu^6$ and (21) gives

$$w(r, z) = \frac{P\mu^4\eta}{4\pi(C_1 + \mu^2C_2)}\left[\frac{(\mu^6+1)}{\{r^2 + z^2\}^{1/2}} - \frac{2}{\{r^2 + \mu^6z^2\}^{1/2}}\right] \tag{23a}$$

where

$$\eta = \{(\mu^3 - 1)(\mu^9 + \mu^6 + 3\mu^3 - 1)\}^{-1}. \tag{23b}$$

Further, in the special case of a Neo-Hookean material $C_2 = 0$ the result (23a) reduces to that given by Woo and Shield [9]. However, the result (23a) is of a neater form than that given by Woo and Shield [9].

5. THE INFINITE PLATE PROBLEM

In this section we consider the axisymmetric flexure of an elastic plate of infinite extent resting on an incompressible elastic halfspace which is finitely deformed by a radial stress in a direction parallel to the plane boundary (Fig. 1). The plate-elastic halfspace interface is assumed to be smooth and capable of sustaining tensile normal tractions. (i.e. We assume that there is no loss of contact between the plate and the elastic halfspace). The infinitesimal flexural deflections of the infinite plate result from an axisymmetric external load $p(r)$. The deflection of the plate, which corresponds to the surface displacement of the elastic halfspace $w(r, 0)$ is denoted

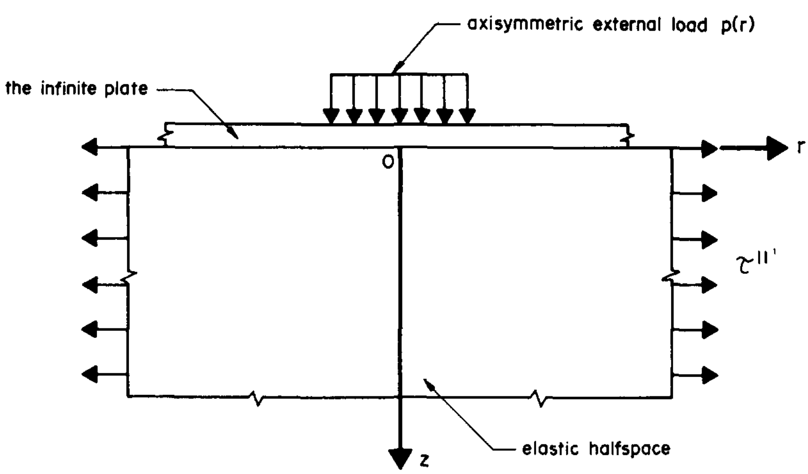

Fig. 1. Axisymmetric loading of the infinite plate.

by $w(r)$. The contact stress at the interface is denoted by $q(r)$. The transformed values of the plate deflection, the contact stress and the external load are defined by

$$[\bar{w}^0(\xi):\bar{q}^0(\xi):\bar{p}^0(\xi)] = \mathcal{H}_0\{[w(r):q(r):p(r)];\xi\}. \tag{24}$$

The differential equation governing flexural deflections of the elastic plate is given by (see e.g. Timoshenko and Woinowsky-Krieger[11]).

$$D\tilde{\nabla}^4 w(r) + q(r) = p(r) \tag{25}$$

where $D(=E_p h^3/12(1-\nu_p^2))$ is the flexural rigidity of the plate, h is its thickness and E_p and ν_p are the elastic constants of the plate material. With reference to the infinite plate problem, the transformed value of the surface displacement $w(r)$, of the initially stressed halfspace is related to the transformed value of the contact stress $q(r)$ by the eqn (20); i.e.

$$\bar{w}^0(\xi) = \frac{\bar{q}^0(\xi)a}{\xi\theta\sqrt{k_2}}\left(\frac{k_2-k_1}{1+k_1}\right). \tag{26}$$

Operating on (25) with the zeroth-order Hankel transform we obtain

$$D\frac{\xi^4}{a^4}\bar{w}^0(\xi) + \bar{q}^0(\xi) = \bar{p}^0(\xi). \tag{27}$$

By substituting for $\bar{q}^0(\xi)$, (27) gives

$$\bar{w}^0(\xi) = \frac{a^4\bar{p}(\xi)}{D\xi\{\xi^3+\Delta\}} \tag{28a}$$

where

$$\Delta = \frac{\theta\sqrt{(k_2)}(1+k_1)a^3}{D(k_2-k_1)}. \tag{28b}$$

The inversion of (28a) yields

$$w(r) = \frac{a^2}{D}\int_0^\infty \frac{\bar{p}^0(\xi)J_0(\xi r/a)}{[\xi^3+\Delta]}\,d\xi. \tag{29a}$$

Similarly from (26) and (28a) it can be shown that the contact stress distribution at the plate-elastic halfspace interface is given by

$$q(r) = \frac{1}{a^2}\int_0^\infty \frac{\xi\Delta\bar{p}^0(\xi)J_0(\xi r/a)}{[\xi^3+\Delta]}\,d\xi. \tag{29b}$$

The flexural moments (M_r, M_θ) and shearing force (Q_r) in the infinite plate can be obtained by making use of (29a) and the expressions

$$\begin{aligned} M_r &= -D\left\{\frac{d^2w}{dr^2} + \frac{\nu_p}{r}\frac{dw}{dr}\right\} \\ M_\theta &= -D\left\{\nu_p\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right\} \\ Q_r &= -D\frac{d}{dr}\{\tilde{\nabla}^2 w\}. \end{aligned} \tag{29c}$$

6 NUMERICAL RESULTS

The expressions developed for the deflection of the finite plate (29a), and the contact stress, (29b), due to the external load $p(r)$ makes no assumptions regarding the manner in which the strain energy function W depends on the invariants I_1 and I_2. For the purposes of discussion and illustration we consider the flexure of the infinite plate subjected to a concentrated force, P, and resting on an initially stressed halfspace with a strain energy function of the Mooney–Rivlin type (22). The small deflection of the infinite plate is thus given by

$$w(r) = \frac{Pa^2}{2\pi D}\int_0^\infty \frac{J_0(\xi r/a)}{[\xi^3 + \Delta^*]}\,d\xi \tag{30}$$

where

$$\Delta^* = \left[\frac{(1+\mu^2\Gamma)(\mu^9+\mu^6+3\mu^3-1)E_s a^3}{3\mu^4(1+\Gamma)(\mu^3+1)D}\right] \tag{31}$$

$$\Gamma = C_2/C_1$$

and $E_s(=6(C_1+C_2))$ is the linear elastic modulus of the incompressible material. The contact stress distribution at the plate-elastic halfspace interface is given by

$$q(r) = \frac{P\Delta^*}{2\pi a^2}\int_0^\infty \frac{\xi J_0(\xi r/a)}{[\xi^3+\Delta^*]}\,d\xi. \tag{32}$$

Taking the limit of (31) as μ approaches unity we obtain

$$\underset{\mu\to 1}{Lt}\,\Delta^* = \frac{2E_s a^3}{3D}. \tag{33}$$

The result (33) when substituted in (30) and (32) gives expressions for the plate deflection and contact stress which are consistent with the solution for the axisymmetric flexure of an infinite plate resting on an unstressed incompressible elastic halfspace (see e.g. Holl[12]).

To illustrate the effects of the initial finite deformation, we present numerical results for the deflection and contact stress at the point of application of the concentrated force P. In this case the values of infinite intergrals (30) and (32) can be evaluated in closed form. We have

$$w(0) = \frac{Pl_0^2}{3\sqrt{3}D}[\Omega]^{2/3}$$
$$q(0) = \frac{P}{3\sqrt{3}\,l_0^2[\Omega]^{2/3}} \tag{34}$$

where

$$\Omega = \left[\frac{2\mu^4(1+\Gamma)(\mu^3+1)}{(1+\mu^2\Gamma)(\mu^9+\mu^6+3\mu^3-1)}\right] \tag{35a}$$

and

$$l_0^3 = \frac{3D}{2E_s} \tag{35b}$$

Since $\Omega = 1$ at $\mu = 1$, the expressions for the plate deflection and the contact stress at the point of application of the concentrated force are expressed as a multiple of the equivalent results for the initially unstressed elastic halfspace. The expression for the plate deflection (34) contains the term $(\mu^9+\mu^6+3\mu^3-1)$ in the denominator of the expression for Ω. Thus the value of $w(0)$ increases without limit as μ approaches a value near to 2/3. This result indicates that when the elastic halfspace is acted upon by a finite radial compression in planes parallel to the bounding surface, the equilibrium becomes unstable for certain critical values of compressive stress. The critical value of μ is identical to that obtained by Green *et al.*[1], Woo and Shield[9] and Beatty

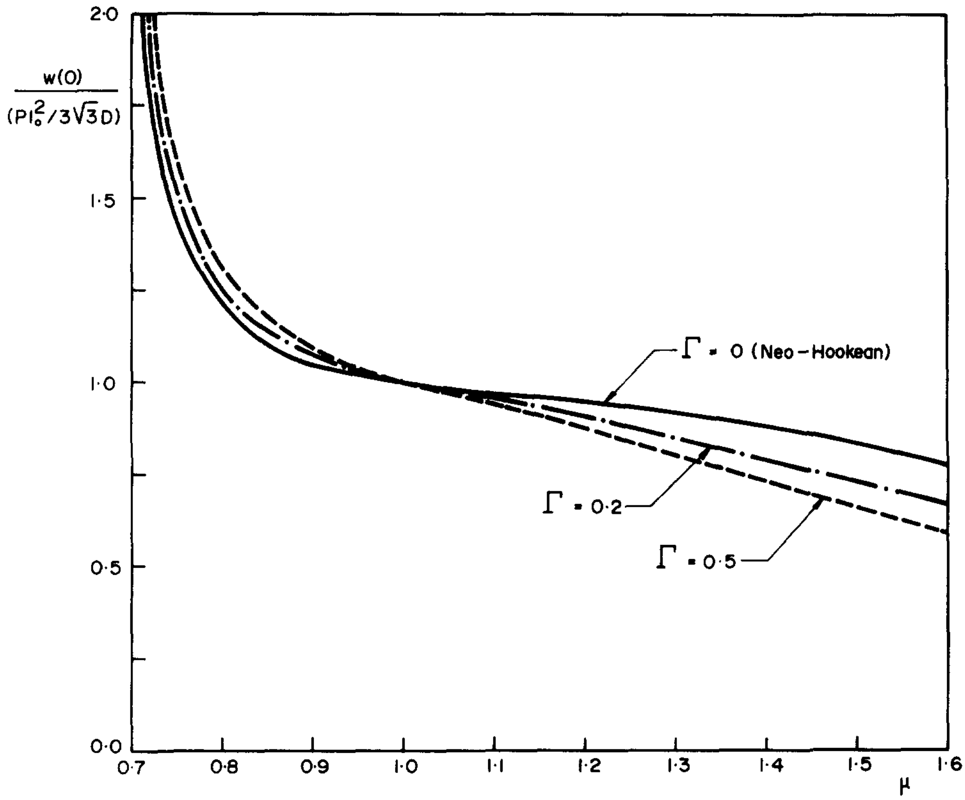

Fig. 2. The variation of plate deflection at the point of application of the concentrated force.

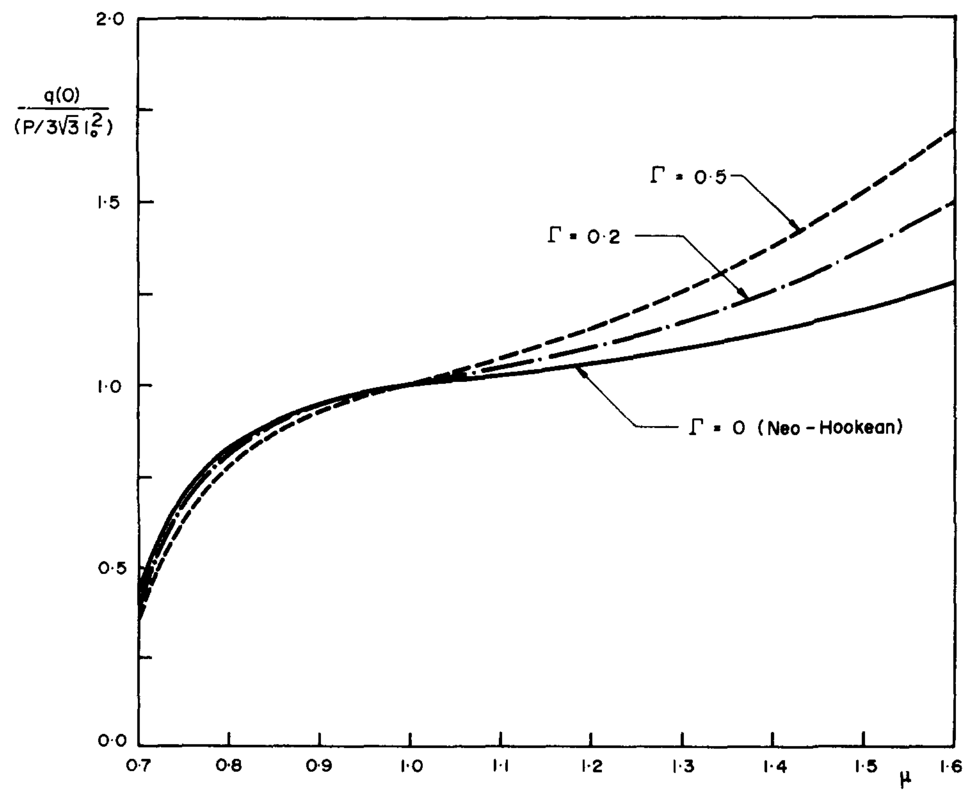

Fig. 3. The variation of contact stress at the plate-elastic halfspace interface at the point of application of the concentrated force.

and Usmani[13] in connection with superposed axisymmetric surface loading of an initially stressed incompressible elastic halfspace with a strain energy function of the Mooney–Rivlin type. The variation of the plate deflection and contact stress at the point of application of the concentrated force are shown in Figs. 2 and 3 respectively. The distribution of $w(r)$ and $q(r)$ for $r > 0$ can be computed by using the non-linear transformation methods for infinite series and integrals derived by Levin[14].

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